Symmetry operators and intertwining operators for the nonlocal Gross–Pitaevskii equation

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Shorttitle: Symmetry operators of the Gross–Pitaevskii equation

Abstract

We consider symmetry properties of an integro-differential multidimensional Gross–Pitaevskii equation with nonlocal cubic nonlinearity in the context of symmetry analysis using the formalism of semiclassical asymptotics. This yields a semiclassically reduced nonlocal Gross–Pitaevskii equation which determines the principal term of a semiclassical asymptotic solution and can be referred to as a nearly linear equation.

Our main result is an approach which allows one to construct a class of symmetry operators for the reduced Gross–Pitaevskii equation. These symmetry operators are determined by linear relations including intertwining operators and additional algebraic conditions.

The basic ideas are illustrated by the examples of a 1D reduced Gross–Pitaevskii equation. The symmetry operators are found explicitly, and the corresponding families of exact solutions generated by the symmetry operators are obtained.

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1 Introduction

Symmetry operators which, by definition, leave the set of solutions of an equation invariant, are of essential importance in symmetry analysis of nonlinear partial differential equations (PDEs). The obvious use of symmetry operators of the equation is generation of new solutions from a known solution.

However, no general approaches are known for regular calculation of symmetry operators for nonlinear equations due to the equations that determine symmetry operators are nonlinear operator equations. Solving them is a complicated mathematical problem which requires special techniques not yet developed.

In addition, to solve the determining equations, we have to define the structure of symmetry operators consistent with the determining equations, but there are no recipes for choosing such a structure. As a consequence, finding the symmetry operators for nonlinear equations in general problem statement is an unrealistic task. Therefore, the Lie group properties of PDEs and related

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structures are studied in symmetry analysis of differential equations [1–7] where the main tool is the symmetry that is generator of the Lie group of symmetry operators.

Modification and further development of the classical Lie group techniques for nonlocal (integro-differential) equations has a long history (see e.g. [8–12]). The idea of current research in this area can be formed on the works [13–21].

The symmetries of PDEs are effectively calculated from linear determining equations and widely used in finding of classes of particular solutions, conservation laws, etc.

At the same time, the construction of a Lie group on a given symmetry by integrating of the Lie system can be performed explicitly only for a special kind of symmetries. Having an algorithm of finding of families of symmetry operators we obtain a possibility to construct Lie groups of symmetry operators and consider structure of the correspondent symmetries.

Note that for linear PDEs, symmetry operators are effectively found from linear determining equations and are widely used in applications such as quantum mechanics (see, e.g., [6,7,22,23] and references therein). This fact leads to the idea to look for a special class of nonlinear equations for which symmetry operators could be calculated using the methods applicable to linear equations.

An example of such a class of nonlinear integro-differential equations (IDEs) with partial derivatives is considered here. We call equations of this class *the nearly linear equations*. The symmetry operators for them can be found by solving linear operator equations (similar to linear PDEs) and additional algebraic equations.

We consider a generalized multidimensional integro-differential Gross–Pitaevskii equation (GPE) with partial derivatives and a nonlocal cubic-nonlinear interaction term of general form. The WKB-Maslov method of semiclassical asymptotics [24, 25] is used to obtain a reduced GPE from the original GPE. The reduced GPE is quadratic in spatial coordinates and derivatives and contains a nonlocal cubic-nonlinear interaction term of special form. This equation belongs to the class of nearly linear equations and determines the principal term of the semiclassical asymptotic solution.

The main result of our work is an approach developed for finding of symmetry operators for the reduced GPE by solving linear operator equations.

The general way for using this approach is illustrated by the example of a one-dimensional reduced GPE whose symmetry operators are found explicitly. With the use of symmetry operators obtained, two families of exact solutions are generated for the reduced GPE.

In the following section, an integro-differential Gross–Pitaevskii equation is introduced and its semiclassical reduction is presented. A method for integrating the reduced GPE is described. We obtain the foundation stones of the method, namely a consistent system and a linear equation associated with the reduced GPE.

In section 2 we propose an approach for finding the class of symmetry operators of the reduced GPE by constructing intertwining operators.

In section 3 the general ideas are illustrated by the example of a one-dimensional GPE of special type. The symmetry operators for this equation are found explicitly and two families of exact solutions are generated with the use of the operators obtained.

2 The nonlocal Gross–Pitaevskii equation and the consistent system

The Gross–Pitaevskii equation and its modifications are widely used in studying coherent matter waves in the Bose–Einstein condensates (BECs) [26–29]. Recent extensions to the BEC studies involve long-range effects in the condensates described by a generalized GPE containing integral terms responsible for nonlocal interactions. We refer to equations of this class as nonlocal GPE (which are also known as the Hartree-type equations in the mathematical literature). The nonlocal BEC models, when used in the theory of BECs with attractive particle interactions, may keep the condensate wave function from collapse and stabilize the solutions in higher dimensions (see, e.g., [30–32], a review [33], and references therein). Nonlocal GPEs also serve as basic equations in models describing many-particle quantum systems, nonlinear optics phenomena [34], collective soliton excitations in atomic chains [35], etc.

Let us write down the nonlocal Gross–Pitaevskii equation as

$$\hat{F}(\Psi)(\vec{x},t) = \{-i\hbar\partial_t + \hat{H}(t) + \varkappa\hat{V}(\Psi)(t)\}\Psi(\vec{x},t) = 0, \tag{1}$$

$$\hat{V}(\Psi)(t) = V(\Psi)(\hat{z},t) = \int\limits_{\mathbb{R}^n} d\vec{y}\,\Psi^*(\vec{y},t)V(\hat{z},\hat{w},t)\Psi(\vec{y},t), \tag{2}$$

where $\Psi(\vec{x},t)$ is a smooth complex scalar function belonging to a complex Schwartz space $\mathbb{S}$ in the space variable $\vec{x} \in \mathbb{R}^n$ at each point in time t. In our consideration, we solve the Cauchy problem

$$\Psi(\vec{x},t)\big|_{t=s} = \psi(\vec{x}) \tag{3}$$

with the initial (at $t = s$) function $\psi(\vec{x}) \in \mathbb{S}$.

The space $\mathbb{S}$ is taken to ensure existence of the Euclidean norm $\|\Psi(t)\| = \sqrt{(\Psi(t),\Psi(t))}$, of the moments of $\Psi(\vec{x},t)$, and of the integral in (2). Here, $(\Phi(t),\Psi(t)) = \int\limits_{\mathbb{R}^n} d\vec{x}\,\Phi^*(\vec{x},t)\Psi(\vec{x},t)$ denotes the Hermitian inner product of functions $\Phi(\vec{x},t)$ and $\Psi(\vec{x},t)$ of the space $\mathbb{S}$; $\Phi^*(\vec{x},t)$ is complex conjugate to $\Phi(\vec{x},t)$.

The linear operators $\hat{H}(t) = H(\hat{z},t)$ and $V(\hat{z},\hat{w},t)$ in (1) are the Hermitian Weyl-ordered functions [36] of time t and of the noncommuting operators

$$\begin{gathered} \hat{z} = (\hat{\vec{p}},\vec{x}) = (-i\hbar\partial/\partial\vec{x},\vec{x}), \\ \hat{w} = (-i\hbar\partial/\partial\vec{y},\vec{y}), \qquad \vec{x},\vec{y} \in \mathbb{R}^n, \end{gathered} \tag{4}$$

with commutators

$$[\hat{z}_k,\hat{z}_j]_- = [\hat{w}_k,\hat{w}_j]_- = i\hbar J_{kj}, \quad [\hat{z}_k,\hat{w}_j]_- = 0, \qquad k,j = \overline{1,2n}, \tag{5}$$

where $[\hat{A},\hat{B}]_- = \hat{A}\hat{B} - \hat{B}\hat{A}$, $J = \|J_{kj}\|_{2n\times 2n}$ is a unit symplectic matrix: $J = \begin{pmatrix} 0 & -\mathbb{I} \\ \mathbb{I} & 0 \end{pmatrix}_{2n\times 2n}$, and $\mathbb{I} = \mathbb{I}_{n\times n}$ is an $n \times n$ identity matrix. We also use shortcut symbols for derivatives: $\partial_t = \partial/\partial t$, $\partial_{\vec{x}} = \partial/\partial\vec{x}$.

From Eq. (1) it follows immediately that the squared norm of a solution $\Psi(\vec{x},t)$ of Eq. (1) is invariable with time, $\|\Psi(t)\|^2 = \|\Psi\|^2 = \|\psi\|^2$.

In the multidimensional case, the GPE (1) with variable coefficients of general form is non-integrable by known methods, such as the Inverse Scattering Transform [37]. Therefore, analytical solutions to this equation can be constructed only approximately. An effective approach to constructing such asymptotic solutions is the method of semiclassical asymptotics at $\hbar \to 0$. Thus, the theory of canonical operator with real phases was developed by Maslov [38, 39] for solving of the Cauchy problem for self-consistent field equations. Spectral problems with singular potentials were considered in [40, 41] (see also [42–45]) and soliton-like solutions of the Hartree-type equation with potentials of special form were constructed [46]. A specific and attractive feature of the nonlocal GPE (1) is that, in the semiclassical approximation, the input GPE is reduced to an equation containing nonlocal terms which can be expressed as a finite number of moments of the unknown function $\Psi(\vec{x},t)$. The reduced equation can be considered as a nearly linear equations. The concept of a nearly linear equation implies that among the solutions of a nonlinear equation there is a subset of solutions regularly depending on a nonlinearity parameter [36].

According to the WKB-Maslov complex germ method [24, 25] applied to a class of trajectory-concentrated functions, the construction of semiclassical asymptotics for Eq. (1) is reduced to an auxiliary problem for an *associated linear* Schrödinger equations.

A solution of the Cauchy problem (1) and (3), asymptotic in a formal small parameter $\hbar$ ($\hbar \to 0$), was constructed accurate to $O(\hbar^{N/2})$ where N is any natural number [47]. The leading term of the asymptotics is found by reducing the GPE (1) to a GPE with a quadratic nonlocal operator

$$\{-i\hbar\partial_t + \hat{H}_q(\hat{z},t) + \varkappa \int\limits_{\mathbb{R}^n} d\vec{y}\,\Psi^*(\vec{y},t)V_q(\hat{z},\hat{w},t)\Psi(\vec{y},t)\}\Psi(\vec{x},t) = 0 \tag{6}$$

where the linear operators $H_q(\hat{z},t)$ and $V_q(\hat{z},\hat{w},t)$ are Hermitian and quadratic in $\hat{z}$, $\hat{w}$:

$$H_q(\hat{z},t) = \frac{1}{2}\langle\hat{z},\mathcal{H}_{zz}(t)\hat{z}\rangle + \langle\mathcal{H}_z(t),\hat{z}\rangle, \tag{7}$$

$$V_q(\hat{z},\hat{w},t) = \frac{1}{2}\langle\hat{z},W_{zz}(t)\hat{z}\rangle + \langle\hat{z},W_{zw}(t)\hat{w}\rangle + \frac{1}{2}\langle\hat{w},W_{ww}(t)\hat{w}\rangle. \tag{8}$$

Here, $\mathcal{H}_{zz}(t)$, $W_{zz}(t)$, $W_{zw}(t)$, and $W_{ww}(t)$ are $2n \times 2n$ matrices; $\mathcal{H}_z(t)$ is a $2n$ vector; the angle brackets $\langle .,. \rangle$ denote the Euclidean inner product of vectors: $\langle \vec{p}, \vec{x} \rangle = \sum\limits_{j=1}^{n} p_j x_j$; $\vec{p}, \vec{x} \in \mathbb{R}^n$; $\langle z, w \rangle = \sum_{j=1}^{2n} z_j w_j$; $z, w \in \mathbb{R}^{2n}$. We shall call equation (6) *a reduced Gross–Pitaevskii equation.*

The reduced GPE given by Eqs. (6), (7), (8) can be integrated explicitly [49,50] and it possess fairly rich symmetries. Analysis of these symmetries can provide a wealth of information about the equation and its solutions.

As Eq. (6) contains a nonlocal nonlinear term, its symmetry properties are of special interest in the symmetry analysis of partial derivatives equations. The matter is that the application of the standard methods of symmetry analysis [1–3,5,6], developed basically for PDEs, to equations different in structure from PDEs presents a number of difficulties: For instance, there are no regular rules for choosing an appropriate structure of symmetries for non-differential equations. The reduced equation (6) allows one to avoid this problem, as the symmetry properties of this equation are closely related to the symmetry of the linear equation associated with the input nonlinear equation.

The key factor in analyzing the symmetries of the nonlinear equation $\hat{F}(\Psi)(\vec{x}, t) = 0$ is the symmetry operator $\hat{A}$ that makes the set of solutions of the equation invariant (see, e.g., [7,22]):

$$\hat{F}(\Psi)(\vec{x}, t) = 0 \Rightarrow \hat{F}(\hat{A}\Psi)(\vec{x}, t) = 0. \tag{9}$$

Generally, it is impossible to find effectively a symmetry operator $\hat{A}$ for a given nonlinear operator $\hat{F}$ by solving the nonlinear operator equation (9). This situation is resolved in the group analysis of differential equations [1–4] where a symmetry $\hat{\sigma}$ (generator of a Lie group of symmetry operators) is the main object of analysis.

The symmetries are determined by the linear operator equation

$$\hat{F}(\Psi)(\vec{x}, t) = 0 \Rightarrow \hat{F}'(\hat{\sigma}\Psi)(\vec{x}, t) = 0. \tag{10}$$

Here, $\hat{F}'(\Psi)$ is the Freshet derivative of $\hat{F}$ calculated for Ψ. For the linear operator $\hat{F}$, we have $\hat{F}' = \hat{F}$ and the symmetry operators being the same as the symmetries.

We assign the GPE (6) to the class of nearly linear equations following the definition given elsewhere [48]: A nearly linear equation determining a function Ψ has the form of a linear partial differential equation with coefficients depending on the moments of the function Ψ. This type of equation can be associated with a consistent system which includes a system of ordinary differential equations (ODEs) describing the evolution of the moments and the reduced GPE.

Using the GPE as an example, we can see that the class of symmetry operators for nearly linear equations can be found by solving the respective determining *linear* operator equations. In this sense, the symmetry properties of nearly linear equations are similar in many respects to those of linear equations.

Let us consider briefly a method for solving the Cauchy problem (3) for the reduced GPE (6) following the scheme descibed elsewhere [47].

We denote the Weyl-ordered symbol of an operator $\hat{A}(t) = A(\hat{z}, t)$ by $A(z, t)$ and define the expectation value for $\hat{A}(t)$ over the state $\Psi(\vec{x}, t)$ as

$$A_\Psi(t) = \frac{1}{\|\Psi\|^2}(\Psi, \hat{A}(t)\Psi) = \frac{1}{\|\Psi\|^2}\int\limits_{\mathbb{R}^n} d\vec{x}\, \Psi^*(\vec{x}, t)\hat{A}(t)\Psi(\vec{x}, t).$$

As $\|\Psi\|^2$ does not depend on time, we have from (6),(7), (8):

$$\begin{aligned} \dot{A}_\Psi(t) = \frac{1}{\|\Psi\|^2}\int\limits_{\mathbb{R}^n} d\vec{x}\, \Psi^*(\vec{x}, t)\Big\{\frac{\partial \hat{A}(t)}{\partial t} + \frac{i}{\hbar}[H_q(\hat{z}, t), \hat{A}(t)]_- + \\ + \frac{i}{\hbar}\int\limits_{\mathbb{R}^n} d\vec{y}\, \Psi^*(\vec{y}, t)[V_q(\hat{z}, \hat{w}, t), \hat{A}(t)]_- \Psi(\vec{x}, t)\Big\}, \end{aligned} \tag{11}$$

where $\dot{A}_\Psi(t) = dA_\Psi(t)/dt$.

We call Eq. (11), similar to the linear Schrödinger equation in quantum mechanics ($\varkappa = 0$ in Eq. (1)), *the Ehrtenfest equation* for GPE (6).

Let $z_\Psi(t) = \big(z_{\Psi\,l}(t)\big)$ and $\Delta^{(2)}_\Psi(t) = \big(\Delta^{(2)}_{\Psi\,kl}(t)\big)$ denote the expectation values over $\Psi(\vec{x},t)$ for the operators

$$\hat{z}_l, \quad \hat{\Delta}^{(2)}_{kl} = \frac{1}{2}\Big(\Delta\hat{z}_k\Delta\hat{z}_l + \Delta\hat{z}_l\Delta\hat{z}_k,\Big), \quad k,l = \overline{1,2n}, \tag{12}$$

respectively. Here, $\Delta\hat{z}_l = \hat{z}_l - (z_\Psi)_l(t)$. We call $z_\Psi(t)$ the first moments and $\Delta^{(2)}_\Psi(t)$ do the second centered moments of $\Psi(\vec{x},t)$.

From (6), (7), (8), and (11) we immediately obtain a dynamical system in matrix notation:

$$\begin{cases} \dot{z}_\Psi = J\{\mathcal{H}_z(t) + [\mathcal{H}_{zz}(t) + \tilde{\varkappa}(W_{zz}(t) + W_{zw}(t))]z_\Psi\}, \\ \dot{\Delta}^{(2)}_\Psi = J[\mathcal{H}_{zz}(t) + \tilde{\varkappa}W_{zz}(t)]\Delta^{(2)}_\Psi - \Delta^{(2)}_\Psi[\mathcal{H}_{zz}(t) + \tilde{\varkappa}W_{zz}(t)]J. \end{cases} \tag{13}$$

We call Eqs. (13) *the Hamilton–Ehrenfest system* (HES) of the second order for GPE (6). The HES is of the second order, as Eqs. (13) contain the first and second moments.

For brevity, we use a shorthand notation for the total set of the first and second moments of $\Psi(\vec{x},t)$:

$$\mathfrak{g}_\Psi(t) = \big(z_\Psi(t), \Delta^{(2)}_\Psi(t)\big). \tag{14}$$

The functions $\mathfrak{g} = \mathfrak{g}_\Psi(t)$ describe phase orbits in the phase space of system (13).

Then the Cauchy problem (3) for the reduced GPE (6) can be written equivalently as

$$\hat{L}(t,\mathfrak{g}_\Psi(t))\Psi(\vec{x},t) = \Big\{ -i\hbar\partial_t + \hat{H}_q(t,\mathfrak{g}_\Psi(t))\Big\}\Psi(\vec{x},t) = 0, \tag{15}$$

$$\begin{aligned} \hat{H}_q(t,\mathfrak{g}_\Psi(t)) = {}& \frac{1}{2}\langle\hat{z},\mathcal{H}_{zz}(t)\hat{z}\rangle + \langle\mathcal{H}_z(t),\hat{z}\rangle + \frac{\tilde{\varkappa}}{2}\langle\hat{z},W_{zz}(t)\hat{z}\rangle + \\ & + \frac{\tilde{\varkappa}}{2}\langle z_\Psi(t),W_{ww}(t)z_\Psi(t)\rangle + \tilde{\varkappa}\langle\hat{z},W_{zw}(t)z_\Psi(t)\rangle + \frac{\tilde{\varkappa}}{2}\,\mathrm{Sp}\,\Big[W_{ww}(t)\Delta^{(2)}_\Psi(t)\Big]\Big\}, \end{aligned} \tag{16}$$

$$\dot{\mathfrak{g}}_\Psi(t) = \Gamma(t,\mathfrak{g}_\Psi(t)), \tag{17}$$

$$\Psi(\vec{x},t)\Big|_{t=s} = \psi(\vec{x}), \qquad \mathfrak{g}_\Psi(t)\Big|_{t=s} = \mathfrak{g}_\psi. \tag{18}$$

Equation (17) is a concise form of HES (13), and $\Gamma\big(t,\mathfrak{g}_\Psi(t)\big)$ designates the rhs of (13).

We call GPE (15) and the correspondent HES (17) *the consistent system* for GPE (6).

The GPE (15) can be assigned to the class of nearly linear equations [48], as the operator (16) of the GPE (15) is a linear partial differential operator with coefficients depending only on the first and second moments $\mathfrak{g}_\Psi(t)$.

The consistent system (15), (17) allows us to reduce the Cauchy problem for the GPE (15) to the Cauchy problem for a linear PDE owing to which the Cauchy problem (18) for HES (17) can be solved independently of Eq. (15).

Let

$$\mathfrak{g}(t,\mathbf{C}) = \big(z(t,\mathbf{C}), \Delta^{(2)}(t,\mathbf{C})\big) \tag{19}$$

be the general solution of HES (17) and $\mathbf{C} = (C_1, C_2, \dots, C_N)$ denote the set of integration constants.

Consider a linear PDE with coefficients depending on parameters $\mathbf{C}$:

$$\hat{L}(t,\mathbf{C})\Phi(\vec{x},t,\mathbf{C}) = \big\{ -i\hbar\partial_t + \hat{H}_q(t,\mathbf{C})\big\}\Phi(\vec{x},t,\mathbf{C}) = 0, \tag{20}$$

where

$$\begin{aligned} \hat{H}_q(t,\mathbf{C}) = {}& \frac{1}{2}\langle\hat{z},\mathcal{H}_{zz}(t)\hat{z}\rangle + \langle\mathcal{H}_z(t),\hat{z}\rangle + \frac{\tilde{\varkappa}}{2}\langle\hat{z},W_{zz}(t)\hat{z}\rangle + \\ & + \tilde{\varkappa}\langle\hat{z},W_{zw}(t)Z(t,\mathbf{C})\rangle + \frac{\tilde{\varkappa}}{2}\langle Z(t,\mathbf{C}),W_{ww}(t)Z(t,\mathbf{C})\rangle + \frac{\tilde{\varkappa}}{2}\,\mathrm{Sp}\,\Big[W_{ww}(t)\Delta^{(2)}(t,\mathbf{C})\Big]. \end{aligned} \tag{21}$$

The operator $\hat{H}_q(t,\mathbf{C})$ (21) of Eq. (20) is obtained from (16) where the general solution $\mathfrak{g}(t,\mathbf{C})$ of the HES (17) stands for the moments $\mathfrak{g}_\Psi(t)$.

We call Eq. (20) *the linear associated equation* (LAE) for GPE (15).

Let $\Phi(\vec{x}, t, \mathbf{C}[\psi])$ denote the solution of the Cauchy problem for LAE (20) with the initial condition

$$\Phi(\vec{x}, t, \mathbf{C}[\psi])\Big|_{t=s} = \psi(\vec{x}), \tag{22}$$

where integration constants $\mathbf{C}$ are replaced by the functionals $\mathbf{C} = \mathbf{C}[\psi]$ determined from the algebraic conditions

$$\mathfrak{g}(t, \mathbf{C})\Big|_{t=s} = \mathfrak{g}_\psi. \tag{23}$$

Then the solution of the Cauchy problem (15), (16) for the GPE (see [47, 50] for details) is

$$\Psi(\vec{x}, t) = \Phi(\vec{x}, t, \mathbf{C}[\psi]). \tag{24}$$

Define $\mathbf{C}[\Psi](t)$ by the condition

$$\mathfrak{g}(t, \mathbf{C}[\Psi](t)) = \mathfrak{g}_\Psi(t). \tag{25}$$

From the uniqueness of the solution of the Cauchy problem for the HES (18) it follows that [50]

$$\mathfrak{g}(t, \mathbf{C}[\Psi](t)) = \mathfrak{g}(t, \mathbf{C}[\psi]). \tag{26}$$

and, hence,

$$\mathbf{C}[\Psi](t) = \mathbf{C}[\psi], \tag{27}$$

i.e., the functionals $\mathbf{C}[\Psi](t)$ are the integrals of Eq. (1).

Also, we have

$$\mathfrak{g}(t, \mathbf{C}[\psi]) = \mathfrak{g}_\psi(t), \tag{28}$$

where $\mathfrak{g}_\psi(t)$ is the solution of HES (17) with the initial condition (23).

From (26) we see, in particular, that the parameters $\mathbf{C}[\Psi](t)$ related to $\Psi(\vec{x}, t)$ do not depend on time, i.e. $\mathbf{C}[\Psi](t) = \mathbf{C}[\psi]$.

When the symbols $H(z, t)$ and $V(z, w, t)$ of the operators (1) and (2) are not quadratic functions of z and w, the operator $\hat{H}(t, \mathbf{C})$ in (21) can be presented as

$$\hat{H}(t, \mathbf{C}) = \mathcal{H}(\hat{z}, t) + \tilde{\varkappa} \sum_{|\nu|=0}^{\infty} \frac{1}{\nu!} \frac{\partial^{|\nu|} V(\hat{z}, w, t)}{\partial w^\nu} \Delta_\nu^{(|\nu|)}(t, \mathbf{C})\Big|_{w=z(t,\mathbf{C})}. \tag{29}$$

Here, $\nu \in \mathbb{Z}_+^{2n}$ is a multi-index

$$\nu = (\nu_1, \ldots, \nu_{2n}), \quad \nu_k \geq 0, \quad k = \overline{1, 2n}, \quad |\nu| = \nu_1 + \ldots + \nu_{2n},$$
$$\nu! = (\nu_1!) \cdots (\nu_{2n}!), \quad \frac{\partial^{|\nu|} W(w)}{\partial w^\nu} = \frac{\partial^{|\nu|} W(w)}{\partial w_1^{\nu_1} \ldots \partial w_{2n}^{\nu_{2n}}},$$

and $\Delta_\nu^{(|\nu|)}$ is a $|\nu|$-order centered moment of the solution $\Psi(\vec{x}, t)$. In other words, $\Delta_\nu^{(|\nu|)}$ is the expectation value of the operator $\hat{\Delta}_\nu^{(|\nu|)}(t) = \Delta_\nu^{(|\nu|)}(\hat{z}, t)$ with the Weyl symbol $\Delta_\nu^{(|\nu|)}(z, t) = (\hat{z} - z_\Psi(t))^\nu$. The associated linear equation (20) is in general not exactly integrable, and to find its approximate solutions, we use the semiclassical asymptotic formalism [47, 52]. Analysis of the LAE of general form involves a great number of additional technical issues associated with the semiclassical approximation that requires a separate study. That is why to illustrate the main ideas of the proposed approach, we should limit our discussion to the case of a quadratic operator for which Eq. (20) is integrable.

Let us now turn to the construction of symmetry operators for Eq. (6). To do this, we use an operator intertwining a pair of LAEs of the form (20), although the symmetry operators can also be found by another methods [49, 50] (see also [51]).

3 Intertwining operator and symmetry operators

According to the definition (9), the nonlinear symmetry operator $\hat{A}(t)$ maps any solution $\Psi(\vec{x},t)$ of Eq. (15) into its another solution:

$$\Psi_A(\vec{x},t) = (\hat{A}(t)\Psi)(\vec{x},t). \tag{30}$$

For $\hat{a} = \hat{A}(t)\big|_{t=s}$ and $\psi(\vec{x})$ given by (3), we can set

$$\psi_a(\vec{x}) = \hat{a}\psi(\vec{x}) = \Psi_A(t)\big|_{t=s}, \tag{31}$$

and $\mathfrak{g}_{\psi_a}$ are the first and second moments of $\psi_a(\vec{x})$ similar to (14).

From the solution of the Cauchy problem for the HES (18) with the initial condition $\mathfrak{g}_\Psi(t)\Big|_{t=s} = \mathfrak{g}_{\psi_a}$, by analogy with (28), we have

$$\mathfrak{g}(t, \mathbf{C}[\psi_a]) = \mathfrak{g}_{\psi_a}(t). \tag{32}$$

According to (24), the solutions $\Psi(\vec{x},t)$ and $\Psi_A(\vec{x},t)$ of the GPE (15) are found as

$$\Psi(\vec{x},t) = \Phi(\vec{x},t,\mathbf{C})\big|_{\mathbf{C}=\mathbf{C}[\psi]} \tag{33}$$

and

$$\Psi_A(\vec{x},t) = \Phi(\vec{x},t,\mathbf{C}')\big|_{\mathbf{C}'=\mathbf{C}'[\psi_a]}, \tag{34}$$

where $\Phi(\vec{x},t,\mathbf{C})$ and $\Phi(\vec{x},t,\mathbf{C}')$ are the solutions of two LAEs of the form (20) with two different sets of integration constants $\mathbf{C}$ and $\mathbf{C}'$, respectively.

To construct the symmetry operator $\hat{A}(t)$, we relate the functions $\Phi(\vec{x},t,\mathbf{C}'[\psi_a])$ and $\Phi(\vec{x},t,\mathbf{C}[\psi])$ by a linear operator $\hat{M}(t,s,\mathbf{C}',\mathbf{C})$ which intertwines the two operators $\hat{L}(t,\mathbf{C}')$ and $\hat{L}(t,\mathbf{C})$ in (20):

$$\hat{L}(t,\mathbf{C}')\hat{M}(t,s,\mathbf{C}',\mathbf{C}) = \hat{R}(t,s,\mathbf{C}',\mathbf{C})\hat{L}(t,\mathbf{C}). \tag{35}$$

Here, the linear operator $\hat{R}(t,s,\mathbf{C}',\mathbf{C})$ is a Lagrangian multiplier, and the initial condition is $\hat{M}(t,s,\mathbf{C}',\mathbf{C})|_{t=s} = \hat{a}$.

From (35) we have that $\Phi(\vec{x},t,\mathbf{C}') = \hat{M}(t,s,\mathbf{C}',\mathbf{C})\Phi(\vec{x},t,\mathbf{C})$ for two arbitrary sets of constants $\mathbf{C}'$ and $\mathbf{C}$, and this is especially true for $\Phi(\vec{x},t,\mathbf{C}'[\psi_a])$ and $\Phi(\vec{x},t,\mathbf{C}[\psi])$ with the constants $\mathbf{C}'[\psi_a]$ and $\mathbf{C}[\psi]$.

To find the operator $\hat{M}(t,s,\mathbf{C}',\mathbf{C})$, we consider a linear intertwining operator $\widehat{\mathcal{D}}(t,s,\mathbf{C}',\mathbf{C})$ for $\hat{L}(t,\mathbf{C}')$ and $\hat{L}(t,\mathbf{C})$ satisfying the conditions

$$\hat{L}(t,\mathbf{C}')\hat{\mathcal{D}}(t,s,\mathbf{C}',\mathbf{C}) = \hat{\mathcal{D}}(t,s,\mathbf{C}',\mathbf{C})\hat{L}(t,\mathbf{C}), \tag{36}$$

$$\widehat{\mathcal{D}}(t,s,\mathbf{C}',\mathbf{C})\Big|_{t=s} = \hat{\mathbb{I}}. \tag{37}$$

We call $\widehat{\mathcal{D}}(t,s,\mathbf{C}',\mathbf{C})$ *the fundamental intertwining operator* for $\hat{L}(t,\mathbf{C}')$ and $\hat{L}(t,\mathbf{C})$. With the use of $\widehat{\mathcal{D}}(t,s,\mathbf{C}',\mathbf{C})$, the operator $\hat{M}(t,s,\mathbf{C}',\mathbf{C})$ of (35) can be presented as

$$\hat{M}(t,s,\mathbf{C}',\mathbf{C}) = \widehat{\mathcal{D}}(t,s,\mathbf{C}',\mathbf{C})\widehat{B}(t,\mathbf{C}). \tag{38}$$

Here, $\widehat{B}(t,\mathbf{C})$ is the linear symmetry operator of LAE (20) satisfying the conditions

$$[\hat{L}(t,\mathbf{C}), \widehat{B}(t,\mathbf{C})] = 0, \quad \widehat{B}(t,\mathbf{C})|_{t=s} = \hat{a}. \tag{39}$$

Hence, given the operator $\widehat{\mathcal{D}}(t,s,\mathbf{C}',\mathbf{C})$ of (36) and the family $\mathcal{B}$ of linear symmetry operators of LAE (39) we can construct the family of nonlinear symmetry operators for the GPE (1):

$$(\hat{A}(t)\Psi)(\vec{x},t) = \widehat{\mathcal{D}}(t,s,\mathbf{C}'[\hat{a}\psi],\mathbf{C}[\Psi](t))\widehat{B}(t,\mathbf{C}[\Psi](t))\Psi(\vec{x},t). \tag{40}$$

Here, $\mathbf{C}'[\hat{a}\psi]$ and $\mathbf{C}[\Psi](=\mathbf{C}[\psi])$ are found from (28) and (25), respectively, and $\widehat{B} \in \mathcal{B}$.

Note that the symmetry operator $\hat{A}(t)$ from Eq. (40) is nonlinear since the operators $\hat{M}$ and $\hat{B}$ depend on the parameters $\mathbf{C}$ which are functionals of the function Ψ.

To find the fundamental intertwining operator $\widehat{\mathcal{D}}(t,s,\mathbf{C}',\mathbf{C})$, we introduce a function $\phi(\vec{x},t,\mathbf{C})$ by the conditions

$$\Phi(\vec{x},t,\mathbf{C}) = \hat{K}(\vec{x},t,s,\mathbf{C})\phi(\vec{x},t,\mathbf{C}), \tag{41}$$
$$\hat{K}(\vec{x},t,s,\mathbf{C}) = \exp[-\langle\vec{X}(t,\mathbf{C}),\nabla\rangle]\exp\Big\{\frac{i}{\hbar}[S(t,\mathbf{C})+\langle\vec{P}(t,\mathbf{C}),\vec{x}\rangle]\Big\}, \tag{42}$$

where $\Phi(\vec{x},t,\mathbf{C})$ is a solution of Eq. (20), the vector $z=Z(t,\mathbf{C})=(\vec{P}(t,\mathbf{C}),\vec{X}(t,\mathbf{C}))$ satisfies Eq. (17), and $S(t,\mathbf{C})$ is a smooth function to be determined.

For $\phi(\vec{x},t,\mathbf{C})$ we have from (20)

$$\begin{aligned}
&\hat{L}_0(\vec{x},t,\mathbf{C})\phi(\vec{x},t,\mathbf{C}) = 0, \qquad (43)\\
&\hat{L}_0(\vec{x},t,\mathbf{C}) = \hat{K}^{-1}(\vec{x},t,s,\mathbf{C})\hat{L}(\vec{x},t,\mathbf{C})\hat{K}(\vec{x},t,s,\mathbf{C}) =\\
&= -i\hbar\partial_t + \langle\dot{\vec{X}}(t,\mathbf{C}),i\hbar\nabla\rangle] + \dot{S}(t,\mathbf{C}) + \langle\dot{\vec{P}}(t,\mathbf{C}),\vec{x}\rangle - \langle\vec{P}(t,\mathbf{C}),\dot{\vec{X}}(t,\mathbf{C})\rangle+\\
&\quad+\frac{1}{2}\langle(\hat{z}+Z(t,\mathbf{C})),\mathcal{H}_{zz}(t)(\hat{z}+Z(t,\mathbf{C}))\rangle + \langle\mathcal{H}_z(t),(\hat{z}+Z(t,\mathbf{C}))\rangle+\\
&\quad+\tilde{\varkappa}\Big[\frac{1}{2}\langle(\hat{z}+Z(t,\mathbf{C})),W_{zz}(t)(\hat{z}+Z(t,\mathbf{C}))\rangle + \langle(\hat{z}+Z(t,\mathbf{C})),W_{zw}(t)Z(t,\mathbf{C})\rangle+\\
&\quad+\frac{1}{2}\langle Z(t,\mathbf{C}),W_{ww}(t)Z(t,\mathbf{C})\rangle + \frac{1}{2}\,\mathrm{Sp}\,\big[W_{ww}(t)\Delta^{(2)}(t,\mathbf{C})\big]\Big].
\end{aligned}$$

Putting

$$S(t,\mathbf{C}) = \int\limits_s^t \Big\{\langle\vec{P}(t,\mathbf{C}),\dot{\vec{X}}(t)\rangle - H_{\varkappa}(t,\mathbf{C})\Big\}dt, \tag{44}$$

where

$$\begin{aligned}
H_{\varkappa}(t,\mathbf{C}) &= \frac{1}{2}\langle Z(t,\mathbf{C}),[\mathcal{H}_{zz}(t)+\tilde{\varkappa}(W_{zz}(t)+2W_{zw}(t)+W_{ww}(t))]Z(t,\mathbf{C})\rangle +\\
&+\langle\mathcal{H}_z(t),Z(t,\mathbf{C})\rangle + \frac{1}{2}\tilde{\varkappa}\mathrm{Sp}[W_{ww}(t)\Delta^{(2)}(t,\mathbf{C})],
\end{aligned}$$

and taking into account (17), we obtain an equation for the function $\phi(\vec{x},t,\mathbf{C})$:

$$\hat{L}_0(\vec{x},t)\phi(\vec{x},t,\mathbf{C}) = 0, \quad \hat{L}_0(\vec{x},t) = -i\hbar\partial_t + \frac{1}{2}\langle\hat{z},\mathcal{H}_{zz}(t)\hat{z}\rangle + \tilde{\varkappa}\frac{1}{2}\langle\hat{z},W_{zz}(t)\hat{z}\rangle. \tag{45}$$

Therefore, the operator $\hat{L}_0(\vec{x},t)$ and the function $\phi(\vec{x},t,\mathbf{C})$ do not depend on the constants $\mathbf{C}$ and equation (36) for the fundamental intertwining operator can be written as

$$\begin{aligned}
&\hat{K}(\vec{x},t,s,\mathbf{C}')\hat{L}_0(\vec{x},t)\hat{K}^{-1}(\vec{x},t,s,\mathbf{C}')\hat{\mathcal{D}}(t,s,\mathbf{C}',\mathbf{C}) =\\
&\quad= \hat{\mathcal{D}}(t,s,\mathbf{C}',\mathbf{C})\hat{K}(\vec{x},t,s,\mathbf{C})\hat{L}_0(\vec{x},t)\hat{K}^{-1}(\vec{x},t,s,\mathbf{C}), \qquad (46)\\
&\widehat{\mathcal{D}}(t,s,\mathbf{C},\mathbf{C}')\Big|_{t=s} = \hat{\mathbb{I}}.
\end{aligned}$$

Hence,

$$\widehat{\mathcal{D}}(t,s,\mathbf{C},\mathbf{C}') = \hat{K}^{-1}(\vec{x},t,s,\mathbf{C}')\widehat{\widehat{\mathcal{D}}}(t,s,\mathbf{C}',\mathbf{C})\hat{K}(\vec{x},t,s,\mathbf{C}), \tag{47}$$

where $\widehat{\widehat{\mathcal{D}}}(t,s,\mathbf{C}',\mathbf{C})$ is a symmetry operator of equation (45), i.e.

$$\begin{aligned}
&\Big[\hat{L}_0(\vec{x},t),\widehat{\widehat{\mathcal{D}}}(t,s,\mathbf{C}',\mathbf{C})\Big]_- = 0, \qquad (48)\\
&\widehat{\widehat{\mathcal{D}}}(t,s,\mathbf{C}',\mathbf{C})\Big|_{t=s} = \widehat{\widehat{\mathcal{D}}}_0(\mathbf{C}',\mathbf{C}).
\end{aligned}$$

Here

$$\begin{aligned}
\widehat{\widehat{\mathcal{D}}}_0(\mathbf{C}',\mathbf{C}) &= \hat{K}(\vec{x},t,s,\mathbf{C}')\hat{K}^{-1}(\vec{x},t,s,\mathbf{C})\Big|_{t=s} = \exp\Big\{\frac{i}{\hbar}\delta S(t,\mathbf{C},\mathbf{C}')\Big\}\\
&\times\exp\Big[\langle\delta\vec{X}_0(\mathbf{C},\mathbf{C}'),\nabla\rangle - \frac{i}{\hbar}\langle\delta\vec{P}_0(\mathbf{C},\mathbf{C}'),\vec{x}\rangle\Big]\exp\Big\{\frac{i}{2\hbar}\langle\delta\vec{X}_0(\mathbf{C},\mathbf{C}'),\delta\vec{P}_0(\mathbf{C},\mathbf{C}')\rangle\Big\}. \qquad (49)
\end{aligned}$$

$$\delta S(t, \mathbf{C}, \mathbf{C}') = S(t, \mathbf{C}') - S(t, \mathbf{C}), \quad \delta Z_0(\mathbf{C}, \mathbf{C}') = \big(\delta\vec{P}_0(\mathbf{C}, \mathbf{C}'), \delta\vec{X}_0(\mathbf{C}, \mathbf{C}')\big),$$
$$\delta\vec{X}_0(\mathbf{C}, \mathbf{C}') = \vec{X}_0(\mathbf{C}') - \vec{X}_0(\mathbf{C}), \quad \delta\vec{P}_0(\mathbf{C}, \mathbf{C}') = \vec{P}_0(\mathbf{C}') - \vec{P}_0(\mathbf{C}). \tag{50}$$

The solution of the Cauchy problem (48) for the operator $\widehat{\widehat{\mathcal{D}}}_0(\mathbf{C}', \mathbf{C})$ can be obtained with standard methods (see, e.g., [22, 52]) as

$$\widehat{\widehat{\mathcal{D}}}(t, s, \mathbf{C}', \mathbf{C}) = \widehat{\widehat{\mathcal{D}}}_0(\mathbf{C}', \mathbf{C}) \exp\Big\{\frac{i}{2\hbar}\langle\delta\vec{X}_0(\mathbf{C}, \mathbf{C}'), \delta\vec{P}_0(\mathbf{C}, \mathbf{C}')\rangle\Big\} \exp\Big\{\hat{b}(t, s, \mathbf{C}', \mathbf{C})\Big\}, \tag{51}$$

where

$$\hat{b}(t, s, \mathbf{C}', \mathbf{C}) = \langle b(t, s, \mathbf{C}', \mathbf{C}), J\hat{z}\rangle, \tag{52}$$

and the $2n$-component vector $b = b(t, s, \mathbf{C}', \mathbf{C})$ is a solution of the Cauchy problem for the system

$$\dot{b} = J[\mathcal{H}_{zz}(t) + \tilde{\varkappa} W_{zz}(t)]b, \qquad b\Big|_{t=s} = \delta Z_0(\mathbf{C}, \mathbf{C}'). \tag{53}$$

Then the symmetry operator $\hat{A}(t)$ for Eq. (15) (or, equivalently, (6)) can be presented as (40), where the intertwining operator $\widehat{\mathcal{D}}(t, s, \mathbf{C}', \mathbf{C})$ is defined by (47) and (51), and $\widehat{B}(t, \mathbf{C})$ is a symmetry operator of LAE (20).

Using the explicit form (47), (51) of the intertwining operator $\widehat{\mathcal{D}}(t, s, \mathbf{C}', \mathbf{C})$ and the operator $\widehat{K}(\vec{x}, t, s, \mathbf{C})$ from (41), we have

$$\Psi_A(\vec{x}, t) = (\hat{A}(t)\Psi)(\vec{x}, t) = \exp\Big\{\frac{i}{\hbar}[S_A(t) + \langle\vec{P}_A(t), \vec{x} - \vec{X}_A(t)\rangle]\Big\}\widehat{B}(\vec{x} + \vec{X}(t) - \vec{X}_A(t), t) \times$$
$$\times \exp\Big\{-\frac{i}{\hbar}[S(t) + \langle\vec{P}(t), \vec{x} - \vec{X}(t)\rangle]\Big\}\Psi(\vec{x} + \vec{X}(t) - \vec{X}_A(t), t), \tag{54}$$

where

$$\widehat{B}(\vec{x}, t) = \widehat{B}(t, \mathbf{C}[\Psi](t)).$$

We note that expression (54) for the symmetry operators is not simple and requires further analysis, but other forms of symmetry operators for the GPE are unknown.

To obtain more simple examples of symmetry operators in explicit form, we consider the 1D case of equations (6), (7), (8).

4 Symmetry operators in 1D case

The reduced 1D GPE (6) reads:

$$\hat{F}(\Psi)(\vec{x}, t) = \{-i\hbar\partial_t + \widehat{H}_q + \varkappa\widehat{V}_q(\Psi)(t)\}\Psi(x, t) = 0, \tag{55}$$
$$\Psi\Big|_{t=0} = \psi(x), \tag{56}$$

where we have used the notation

$$\widehat{H}_q = \frac{1}{2}\left(\mu\hat{p}^2 + \rho(x\hat{p} + \hat{p}x) + \sigma x^2\right), \quad \widehat{V}_q[\Psi] = \frac{1}{2}\int_{-\infty}^{+\infty} dy\,\left(ax^2 + 2bxy + cy^2\right)|\Psi(y)|^2,$$

$\hat{p} = -i\hbar\partial/\partial x$; a, b, and c are the real parameters of the nonlocal operator $\widehat{V}[\Psi]$; μ, σ, ρ are the parameters of the linear operator $\widehat{H}_q$; $x, y \in \mathbb{R}^1$.

The Hamilton–Ehrenfest system (17) for the first-order moments becomes [54]

$$\begin{cases}\dot{p} = -\rho p - \sigma_0 x,\\ \dot{x} = \mu p + \rho x,\end{cases} \tag{57}$$

and for the second-order moments with $\Delta_{21}^{(2)} = \Delta_{12}^{(2)}$ we have

$$\begin{cases}\dot{\Delta}_{11}^{(2)} = -2\rho\Delta_{11}^{(2)} - 2\tilde{\sigma}\Delta_{21}^{(2)},\\ \dot{\Delta}_{21}^{(2)} = \mu\Delta_{11}^{(2)} - \tilde{\sigma}\Delta_{22}^{(2)},\\ \dot{\Delta}_{22}^{(2)} = 2\mu\Delta_{21}^{(2)} + 2\rho\Delta_{22}^{(2)},\end{cases} \tag{58}$$

where

$$\sigma_0 = \sigma + \tilde{\varkappa}(a+b), \quad \tilde{\sigma} = \sigma + \tilde{\varkappa}a.$$

We introduce the notation

$$\bar{\Omega} = \sqrt{\sigma_0\mu - \rho^2}, \qquad \Omega = \sqrt{\tilde{\sigma}\mu - \rho^2} \tag{59}$$

and assume that $\bar{\Omega}^2 = \sigma_0\mu - \rho^2 > 0$. This implies that the ions do not leave the Paul trap. Indeed, in this case, the general solution of system (57) is

$$\begin{aligned} &X(t,\mathbf{C}) = C_1 \sin\bar{\Omega}t + C_2\cos\bar{\Omega}t, \\ &P(t,\mathbf{C}) = \frac{1}{\mu}(\bar{\Omega}C_1 - \rho C_2)\cos\bar{\Omega}t - \frac{1}{\mu}(\bar{\Omega}C_2 + \rho C_1)\sin\bar{\Omega}t; \end{aligned} \tag{60}$$

and all solutions of system (57) are localized.

Assume that the wave packets that describe the evolution of ions by Eq. (55) do not spread. This takes place if $\Omega^2 = \tilde{\sigma}\mu - \rho^2 > 0$.

For system (58) we have

$$\begin{aligned} &\Delta_{22}^{(2)}(t,\mathbf{C}) = C_3\sin 2\Omega t + C_4\cos 2\Omega t + C_5, \\ &\Delta_{21}^{(2)}(t,\mathbf{C}) = \frac{1}{\mu}(\Omega C_3 - \rho C_4)\cos 2\Omega t - \frac{1}{\mu}(\Omega C_4 + \rho C_3)\sin 2\Omega t - \frac{\rho}{\mu}C_5, \\ &\Delta_{11}^{(2)}(t,\mathbf{C}) = \frac{1}{\mu^2}\big((\rho^2 - \Omega^2)C_3 + 2\rho\Omega C_4\big)\sin 2\Omega t + \\ &\qquad + \frac{1}{\mu^2}\big((\rho^2 - \Omega^2)C_4 - 2\rho\Omega C_3\big)\cos 2\Omega t + \frac{\bar{\sigma}}{\mu}C_5, \end{aligned} \tag{61}$$

and all solutions of system (58) are also localized. Here, $\mathbf{C} = (C_1,\dots,C_5)$, and C_l, $l = \overline{1,5}$, are arbitrary integration constants.

The associated linear equation (20) in the 1D case is

$$\begin{aligned} &\hat{L}(x,t,\mathbf{C})\Phi = 0, \\ &\hat{L}(x,t,\mathbf{C}) = -i\hbar\partial_t + \frac{\mu\hat{p}^2}{2} + \frac{\tilde{\sigma}x^2}{2} + \frac{\rho(x\hat{p} + \hat{p}x)}{2} + \\ &\qquad + \tilde{\varkappa}bxX(t,\mathbf{C}) + \tilde{\varkappa}\frac{c}{2}\big[X^2(t,\mathbf{C}) + \Delta_{22}^{(2)}(t,\mathbf{C})\big]. \end{aligned} \tag{62}$$

We can immediately verify that for the associated linear equation (62) we can construct the following set of symmetry operators linear in x and $\hat{p}$:

$$\hat{a}(t,\mathbf{C})) = \frac{1}{\sqrt{2\hbar}}\big[C(t)\big(\hat{p} - P(t,\mathbf{C})\big) - B(t)\big(x - X(t,\mathbf{C})\big)\big], \tag{63}$$

$$\hat{a}^+(t,\mathbf{C})) = \frac{1}{\sqrt{2\hbar}}\big[C^*(t)\big(\hat{p} - P(t,\mathbf{C})\big) - B^*(t)\big(x - X(t,\mathbf{C})\big)\big]. \tag{64}$$

Here the functions $B(t)$ and $C(t)$ are solutions of the linear Hamiltonian system

$$\begin{cases} \dot{B} = -\rho B - \tilde{\sigma}C, \\ \dot{C} = \mu B + \rho C. \end{cases} \tag{65}$$

The Cauchy matrix $\mathcal{X}(t)$ for the system (65) can easily be found as

$$\mathcal{X}(t) = \begin{pmatrix} \cos\Omega t - \dfrac{\rho}{\Omega}\sin\Omega t & -\dfrac{1}{\mu\Omega}(\Omega^2 + \rho^2)\sin\Omega t \\ \dfrac{\mu}{\Omega}\sin\Omega t & \cos\Omega t + \dfrac{\rho}{\Omega}\sin\Omega t \end{pmatrix}, \quad \mathcal{X}(t)\Big|_{t=0} = \mathbb{I}_{2\times 2}. \tag{66}$$

The set of solutions normalized by the condition

$$B(t)C^*(t) - C(t)B^*(t) = 2i \tag{67}$$

can be written as

$$B(t) = e^{i\Omega t}\frac{(-\rho + i\Omega)}{\sqrt{\Omega\mu}}, \qquad C(t) = e^{i\Omega t}\sqrt{\frac{\mu}{\Omega}}.$$

Equation (67) results in the following commutation relations for the symmetry operators (63) and (64):

$$\big[\hat{a}(t,\mathbf{C}), \hat{a}^+(t,\mathbf{C})\big] = 1.$$

For the function ϕ given by (41) in 1D case, we obtain

$$\Phi(x,t,\mathbf{C}) = \hat{K}(\vec{x},t,s,\mathbf{C})\phi(\vec{x},t), \tag{68}$$

$$\hat{K}(x,t,\mathbf{C}) = \exp[-X(t,\mathbf{C})\partial_x]\exp\Big\{\frac{i}{\hbar}[S(t,\mathbf{C}) + P(t,\mathbf{C})x]\Big\}, \tag{69}$$

where

$$S(t,\mathbf{C}) = \int\limits_0^t \Big\{P(t,\mathbf{C})\dot{X}(t) - H_\varkappa(t,\mathbf{C})\Big\}dt, \tag{70}$$

$$H_\varkappa(t,\mathbf{C}) = \frac{\mu}{2}P^2(t,\mathbf{C}) + \frac{1}{2}X^2(t,\mathbf{C})\big[\sigma_0 + \tilde{\varkappa}(b+c)\big] + \rho P(t,\mathbf{C})X(t,\mathbf{C}) + \tilde{\varkappa}\frac{c}{2}\Delta_{22}^{(2)}(t,\mathbf{C}).$$

From (45) we find

$$\hat{L}_0(x,t)\phi = 0, \quad \hat{L}_0(x,t) = -i\hbar\partial_t + \frac{\mu\hat{p}^2}{2} + \frac{(\sigma + \tilde{\varkappa}a)x^2}{2} + \frac{\rho(x\hat{p} + \hat{p}x)}{2}. \tag{71}$$

Then the symmetry operator $\hat{A}(t)$ (47) for Eq. (55) can be presented as

$$(\hat{A}(t)\Psi)(x,t) = \widehat{\mathcal{D}}(t,\mathbf{C}[\hat{a}\psi],\mathbf{C}[\Psi](t))\widehat{B}(t,\mathbf{C}[\Psi](t))\Psi(x,t), \tag{72}$$

where $\widehat{B}(t,\mathbf{C})$ is the symmetry operator of the associated linear equation (62). The intertwining operator $\widehat{\mathcal{D}}(t,\mathbf{C}',\mathbf{C})$ can be presented as

$$\widehat{\mathcal{D}}(t,\mathbf{C},\mathbf{C}') = \hat{K}^{-1}(x,t,\mathbf{C}')\widehat{\widehat{\mathcal{D}}}(t,\mathbf{C}',\mathbf{C})\hat{K}(x,t,\mathbf{C}), \tag{73}$$

$$\begin{aligned}\widehat{\widehat{\mathcal{D}}}(t,\mathbf{C}',\mathbf{C}) &= \exp\Big\{\frac{i}{\hbar}[S(t,\mathbf{C}') - S(t,\mathbf{C})]\Big\}\exp\Big\{\hat{b}(t,\mathbf{C}',\mathbf{C})\Big\}\times \\ &\times\exp\Big\{i\frac{C_2' - C_2}{2\hbar\mu}\Big(\bar{\Omega}(C_1' - C_1) - \rho(C_2' - C_2)\Big)\Big\},\end{aligned} \tag{74}$$

where

$$\hat{b}(t,\mathbf{C}',\mathbf{C}) = b_x(t,\mathbf{C}',\mathbf{C})\hat{p} - b_p(t,\mathbf{C}',\mathbf{C})x = \langle b(t), \hat{z}\rangle, \tag{75}$$

and the vector $b(t)$ is defined by

$$b(t,\mathbf{C}',\mathbf{C}) = \begin{pmatrix} b_p(t,\mathbf{C}',\mathbf{C}) \\ b_x(t,\mathbf{C}',\mathbf{C})\end{pmatrix} = \frac{1}{\mu}\mathcal{X}(t)\begin{pmatrix} \mu(C_2' - C_2) \\ \bar{\Omega}(C_1' - C_1) - \rho(C_2' - C_2)\end{pmatrix}.$$

The matrix $\mathcal{X}(t)$ is given by (66).

The symmetry operator $\hat{A}(t)$ from (72) of nonlinear equation (55) has the structure of a linear pseudodifferential operator whose parameters are functionals of the function on which the operator acts.

Therefore, the explicit form of the operator $\hat{A}(t)$ is determined not only by the symmetry operator of the associated linear equation $\widehat{B}(t,\mathbf{C})$, but also by the function $\Psi(x,t)$. Note that for some values of the parameters (more precisely, for the function $\Psi(x,t)$ that define them) the pseudodifferential operator becomes a differential one.

We set

$$\widehat{B}(t,\mathbf{C}) = \widehat{B}_\nu(t,\mathbf{C}) = \frac{1}{\sqrt{\nu!}}\Big[\hat{a}^+(t,\mathbf{C})\Big]^\nu, \quad \nu \in \mathbb{Z}_+, \tag{76}$$

where the operators $\hat{a}^+(t,\mathbf{C})$ are defined in (63).

Substituting (76) in (72), we obtain the symmetry operator, which we denote by $\hat{A}_\nu(t)$.

With the use of a stationary solution of the Hamilton–Ehrenfest system (57), (58), we simplify the symmetry operators (63), (64) and generate a countable set of explicit solutions of the 1D GPE (55).

A stationary solution of Eqs. (57), (58) is obtained from the general solution (60), (61) if we take integration constants as $\mathbf{C}=\mathbf{C}^0=(C_1^0,\dots,C_5^0)$, $C_1^0=C_2^0=C_3^0=C_4^0=0$, C_5^0 is an arbitrary real constant. The stationary solution is

$$X(t,\mathbf{C})=P(t,\mathbf{C})=0,\quad \Delta_{22}^{(2)}(t,\mathbf{C})=C_5^0,\quad \Delta_{21}^{(2)}(t,\mathbf{C})=-\frac{\rho}{\mu}C_5^0,\quad \Delta_{11}^{(2)}(t,\mathbf{C})=\frac{\bar\sigma}{\mu}C_5^0. \tag{77}$$

Substituting (77) in (62), we obtain the associated linear equation

$$\hat L(x,t,\mathbf{C}^0)\Phi=0,\quad \hat L(x,t,\mathbf{C}^0)=\Big[-i\hbar\partial_t+\frac{\mu\hat p^2}{2}+\frac{(\sigma+\tilde\varkappa a)x^2}{2}+\frac{\rho(x\hat p+\hat p x)}{2}+\tilde\varkappa\frac{c}{2}C_5^0\Big]. \tag{78}$$

The operator $\hat K(\vec x,t,\mathbf{C})=\hat K(\vec x,t,\mathbf{C}^0)$ from (69) is the operator of multiplication by the function

$$\hat K(\vec x,t,\mathbf{C}^0)=\exp\Big\{-\frac{i}{2\hbar}\tilde\varkappa cC_5^0t\Big\}.$$

The linear operators (63) and (64) then become

$$\hat a(t,\mathbf{C}^0)=\frac{1}{\sqrt{2\hbar}}\big[C(t)\hat p-B(t)x\big],\quad \hat a^+(t,\mathbf{C}^0)=\frac{1}{\sqrt{2\hbar}}\big[C^*(t)\hat p-B^*(t)x\big]; \tag{79}$$

they are symmetry operators for Eq. (78).

The function

$$\Phi_0(x,t,\mathbf{C}^0)=\Big(\frac{1}{\pi\hbar}\Big)^{1/4}\Big(\frac{\Omega}{\mu}\Big)^{1/4}\exp\Big\{-\frac{i}{2\hbar}\frac{\rho}{\mu}x^2-\frac{1}{2\hbar}\frac{\Omega}{\mu}x^2\Big\}\exp\Big\{-\frac{i}{2}\Omega t-\frac{i}{2\hbar}\tilde\varkappa cC_5^0t\Big\} \tag{80}$$

is immediately verified to be solution of Eq. (78).

Upon direct substitution, we see that for the function (80), Eqs. (22), (23), which determine the functionals $\mathbf{C}[\Psi](t)$, become

$$\begin{aligned}
&X(0,\mathbf{C})=x_\psi=0,\quad P(0,\mathbf{C})=p_\psi=0,\\
&\Delta_{22}^{(2)}(0,\mathbf{C})=(\Delta_{22}^{(2)})_\psi=\frac{\hbar}{2}\big|C(0)\big|^2=\frac{\hbar\mu}{2\Omega},\\
&\Delta_{11}^{(2)}(t,\mathbf{C})=(\Delta_{11}^{(2)})_\psi=\frac{\hbar}{2}\big|B(0)\big|^2=\frac{\hbar(\varrho^2+\Omega^2)}{2\Omega\mu},\\
&\Delta_{12}^{(2)}(0,\mathbf{C})=(\Delta_{12}^{(2)})_\psi=\frac{\hbar}{4}\big[B(0)C^*(0)+B^*(0)C(0)\big]=-\frac{\hbar\varrho}{2\Omega},\\
&\psi(x)=\Phi_0(x,0,\mathbf{C}^0).
\end{aligned} \tag{81}$$

From (81) and (77) it follows that $C_5^0=(\hbar\mu/2\Omega)$. From (33) and (80) we find a special solution $\Psi_0(x,t)$ of the GPE (55):

$$\begin{aligned}
\Psi_0(x,t)&=\Phi_0(x,t,\mathbf{C}^0)\Big|_{C_5^0=(\hbar\mu/2\Omega)}=\\
&=\Big(\frac{1}{\pi\hbar}\Big)^{1/4}\Big(\frac{\Omega}{\mu}\Big)^{1/4}\exp\Big\{-\frac{i}{2\hbar}\frac{\rho}{\mu}x^2-\frac{1}{2\hbar}\frac{\Omega}{\mu}x^2\Big\}\exp\Big\{-\frac{i}{2}\Omega t-\frac{i\mu}{4\Omega}\tilde\varkappa ct\Big\}.
\end{aligned} \tag{82}$$

The symmetry operators (76), (79) generate from (80) the solutions of the associated linear equation (78) that constitute a Fock basis in the space $L_2(\mathbb{R}$:

$$\begin{aligned}
\Phi_\nu(x,t,\mathbf{C}^{0\prime})&=\widehat B_\nu(t,\mathbf{C}^{\mathbf{0}\prime})\Phi_0(x,t,\mathbf{C}^{0\prime})=\frac{1}{\sqrt{\nu!}}\Big[\hat a^+(t,\mathbf{C}^{0\prime})\Big]^\nu\Phi_0(x,t,\mathbf{C}^{0\prime})=\\
&=\frac{i^\nu}{\sqrt{\nu!}}\Big(\frac{1}{\sqrt2}\Big)^\nu H_\nu\Big(\sqrt{\frac{\Omega}{\hbar\mu}}x\Big)\Phi_0(x,t,\mathbf{C}^{0\prime})\exp\Big\{-i\Omega\nu t\Big\},\quad \nu\in\mathbb{Z}_+,
\end{aligned} \tag{83}$$

where $H_\nu(\zeta)$ are the Hermite polynomials [55]

$$H_\nu(\zeta) = \left(2\zeta - \frac{d}{d\zeta}\right)^\nu \cdot 1.$$

The operator (47), intertwining the operators $\hat{L}(x,t,\mathbf{C}^0)$ and $\hat{L}(x,t,\mathbf{C}^{0\prime})$ of the form (78), reads

$$\widehat{\mathcal{D}}(t,\mathbf{C}^0,\mathbf{C}^{0\prime}) = \exp\Big\{\frac{i}{2\hbar}\tilde{\varkappa}c[C_5^0 - C_5^{0\prime}]t\Big\}. \tag{84}$$

Equations (23) that determine the functionals $\mathbf{C}[\Psi_\nu](t)$ for the functions (83) are written as

$$\begin{aligned}
&X(t,\mathbf{C}) = x_{\psi_a} = 0, \quad P(0,\mathbf{C}) = p_{\psi_a} = 0,\\
&\Delta_{22}^{(2)}(0,\mathbf{C}) = (\Delta_{22}^{(2)})_{\psi_a} = \frac{\hbar}{2}(2\nu+1)\big|C(0)\big|^2 = \frac{\hbar\mu}{2\Omega}(2\nu+1),\\
&\Delta_{11}^{(2)}(0,\mathbf{C}) = (\Delta_{11}^{(2)})_{\psi_a} = \frac{\hbar}{2}\big|B(0)\big|^2(2\nu+1) = \frac{\hbar(\varrho^2+\Omega^2)}{2\Omega\mu}(2\nu+1),\\
&\Delta_{12}^{(2)}(0,\mathbf{C}) = (\Delta_{12}^{(2)})_{\psi_a} = \frac{\hbar}{4}\big[B(0)C^*(0) + B^*(0)C(0)\big](2\nu+1) = -\frac{\hbar\varrho}{2\Omega}(2\nu+1),\\
&\psi_a(x) = \Phi_\nu(x,0,\mathbf{C}^{0\prime}) = \widehat{B}_\nu(0,\mathbf{C}^{\mathbf{0}\prime})\psi(x).
\end{aligned} \tag{85}$$

Here we have used the standard properties of Hermite polynomials [55]. Taking into account (77), we find from (85) that $C_5^{0\prime} = (\hbar\mu/\Omega)(\nu+1/2)$.

Then the symmetry operator $\hat{A}_\nu(t)$ (72) transforms the solution $\Psi_0(x,t)$ of (82) into solution $\Psi_\nu(x,t)$ of the nonlinear GPE (55) according to the following relation:

$$\begin{aligned}
&\Psi_\nu(x,t) = (\hat{A}_\nu(t)\Psi_0)(x,t) =\\
&= \widehat{\mathcal{D}}(t,\mathbf{C}^0,\mathbf{C}^{0\prime})\frac{1}{\sqrt{\nu!}}\Big[\hat{a}^+(t,\mathbf{C}^0)\Big]^\nu\Big|_{C_5^0=(\hbar\mu/2\Omega),C_5^{0\prime}=(\hbar\mu/\Omega)(\nu+1/2)}\Psi_0(x,t) =\\
&= \Big(\frac{1}{\sqrt{2}}\Big)^\nu\Big(\frac{1}{\pi\hbar}\Big)^{1/4}\Big(\frac{\Omega}{\mu}\Big)^{1/4}\exp\Big\{-\frac{i}{2\hbar}\frac{\rho}{\mu}x^2 - \frac{1}{2\hbar}\frac{\Omega}{\mu}x^2\Big\}\times\\
&\times H_\nu\bigg(\sqrt{\frac{\Omega}{\hbar\mu}}x\bigg)\exp\Big\{-i\Big(\nu+\frac{1}{2}\Big)\Big(\frac{\tilde{\varkappa}c\mu}{2\Omega}+\Omega\Big)t\Big\}.
\end{aligned} \tag{86}$$

Functions (86) constitute a countable set of particular solutions of Eq. (55) which are generated from $\Psi_0(x,t)$ by the nonlinear symmetry operator $\hat{A}_\nu(t)$.

Assume that $\mathbf{C} = \mathbf{C}^1 = (C_1^1, C_2^1, 0, 0, C_5^1)$. This choice of the constants yields the following expression phase orbit (60), (61):

$$\begin{aligned}
&X(t,C_1^1,C_2^1) = C_1^1\sin\bar{\Omega}t + C_2^1\cos\bar{\Omega}t,\\
&P(t,C_1^1,C_2^1) = \frac{1}{\mu}(\bar{\Omega}C_1^1 - \rho C_2^1)\cos\bar{\Omega}t - \frac{1}{\mu}(\bar{\Omega}C_2^1 + \rho C_1^1)\sin\bar{\Omega}t,\\
&\Delta_{22}^{(2)}(t,\mathbf{C}^1) = C_5^1, \quad \Delta_{21}^{(2)}(t,\mathbf{C}^1) = -\frac{\rho}{\mu}C_5^1, \quad \Delta_{11}^{(2)}(t,\mathbf{C}^1) = \frac{\bar{\sigma}}{\mu}C_5^1.
\end{aligned} \tag{87}$$

Consider the action of the operator $\hat{A}_0(t)$ entering in (72), (76) on the functions (86).

Let us write operator (47), intertwining the operators $L(x,t,\mathbf{C}^0)$ and $L(x,t,\mathbf{C}'^0)$, given by (78), as

$$\widehat{\mathcal{D}}(t,\mathbf{C},\mathbf{C}^0) = \exp[-X(t,\mathbf{C})\partial_x]\exp\Big\{\frac{i}{\hbar}[S(t,\mathbf{C}) - \frac{1}{2}\tilde{\varkappa}cC_5^0t + P(t,\mathbf{C})x]\Big\}\widehat{\widehat{\mathcal{D}}}(t,\mathbf{C},\mathbf{C}^0), \tag{88}$$

where

$$\widehat{\widehat{\mathcal{D}}}(t,\mathbf{C},\mathbf{C}^0) = \widehat{\widehat{\mathcal{D}}}_0(\mathbf{C},\mathbf{C}^0)\exp\Big\{+\frac{i}{2\hbar\mu}C_2\big(\tilde{\Omega}C_1 - \rho C_2\big)\Big\}\exp\Big\{\hat{b}(t,\mathbf{C},\mathbf{C}^0)\Big\}, \tag{89}$$

$$\hat{b}(t,\mathbf{C},\mathbf{C}^0) = b_x(t,\mathbf{C},\mathbf{C}^0)\hat{p} - b_p(t,\mathbf{C},\mathbf{C}^0)x = \langle b(t),\hat{z}\rangle, \tag{90}$$

and a vector $b(t)$ is defined by the expression

$$b(t,\mathbf{C},\mathbf{C}^0)=\begin{pmatrix}b_p(t,\mathbf{C},\mathbf{C}^0)\\ b_x(t,\mathbf{C},\mathbf{C}^0)\end{pmatrix}=\frac{1}{\mu}\mathcal{X}t)\begin{pmatrix}\mu C_2\\ \tilde{\Omega}C_1-\rho C_2\end{pmatrix}.$$

The matrix $\mathcal{X}(t)$ is given by (66).

Let us construct a nonlinear symmetry operator $\hat{A}(t,\alpha)$, corresponding to the nonstationary phase orbit (60), (61). The operator $\hat{A}(t,\alpha)$ maps the nonstationary solution of equation (55), $\Psi_\nu(x,t)$ (86), into another nonstationary solution of this equation, $\widetilde{\Psi}_\nu(x,t)$. Consider the shift operator

$$\widehat{B}(t,\mathbf{C}^1)=\widehat{B}(t,\alpha,\mathbf{C}^1)=\exp\{\alpha\hat{a}^+(t)-\alpha^*\hat{a}(t)\},\quad \alpha\in\mathbb{C}, \tag{91}$$

where the operators $\hat{a}(t,\mathbf{C})$ and $\hat{a}^+(t,\mathbf{C})$ are defined by expressions (63), (64). Operator (91) is substituted in (72) for the symmetry operator $\widehat{B}(t,\mathbf{C})$.

Let us write the operator $\widehat{B}(t,\alpha,\mathbf{C}^1)$ entering in (91) as

$$\widehat{B}(t,\alpha,\mathbf{C}^1)=\exp\{\beta(t)\hat{p}+\gamma(t)x\}=\exp\Big\{-\frac{i\hbar}{2}\beta(t)\gamma(t)\Big\}\exp\{\gamma(t)x\}\exp\{\beta(t)\hat{p}\}, \tag{92}$$

where

$$\beta(t)=\frac{1}{\sqrt{2\hbar}}\big[C^*(t)\alpha-C(t)\alpha^*\big],\qquad \gamma(t)=\frac{1}{\sqrt{2\hbar}}\big[B(t)\alpha^*-B^*(t)\alpha\big].$$

Thus, we have

$$\begin{aligned}&\psi_a(x)=\widehat{B}(0,\alpha,\mathbf{C}^1)\Phi_\nu(x,0,\mathbf{C}^{0\prime})=\\&\quad=\exp\Big\{-\frac{i\hbar}{2}\beta(0)\gamma(0)\Big\}\exp\{\gamma(0)x\}\Phi_\nu(x-i\hbar\beta(0),0,\mathbf{C}^{0\prime}),\\&\Phi_\nu(x-i\hbar\beta(0),0,\mathbf{C}^{0\prime})=\\&\quad=\frac{i^\nu}{\sqrt{\nu!}}\Big(\frac{1}{\sqrt{2}}\Big)^\nu H_\nu\bigg(\sqrt{\frac{\Omega}{\hbar\mu}}\Big[x-i\hbar\beta(0)\Big]\bigg)\Phi_0(x-i\hbar\beta(0),0,\mathbf{C}^{0\prime}),\\&\Phi_0(x-i\hbar\beta(0),0,\mathbf{C}^{0\prime})=\\&\quad=\Phi_0(x,0,\mathbf{C}^{0\prime})\exp\Big\{\Big[\Big(\frac{-\rho+i\Omega}{\mu}\Big)x\beta(0)-i\frac{\hbar}{2}\Big(\frac{-\rho+i\Omega}{\mu}\Big)\beta^2(0)\Big]\Big\}.\end{aligned} \tag{93}$$

Let us note that

$$\begin{aligned}&\gamma(0)=\frac{1}{\sqrt{2\hbar}}\big[-B^*(0)\alpha+B(0)\alpha^*\big]=\frac{1}{\sqrt{2\hbar}}\Big[\frac{(\rho+i\Omega)}{\sqrt{\Omega\mu}}\alpha+\frac{(-\rho+i\Omega)}{\sqrt{\Omega\mu}}\alpha^*\Big]=i\frac{\sqrt{2}}{\sqrt{\hbar\Omega\mu}}\big[\rho\alpha_2+\Omega\alpha_1\big],\\&\beta(0)=\frac{1}{\sqrt{2\hbar}}\big[C^*(0)\alpha-C(0)\alpha^*\big]=\frac{1}{\sqrt{2\hbar}}\sqrt{\frac{\mu}{\Omega}}(\alpha-\alpha^*)=i\sqrt{\frac{2\mu}{\hbar\Omega}}\alpha_2,\\&\gamma(0)+\frac{-\rho+i\Omega}{\mu}\beta(0)=\frac{1}{\sqrt{2\hbar}}\Big[-B^*(0)\alpha+B(0)\alpha^*+\frac{-\rho+i\Omega}{\mu}(C^*(0)\alpha-C(0)\alpha^*)\Big]=\\&\quad=\frac{1}{\sqrt{2\hbar\Omega\mu}}\big[(\rho+i\Omega)\alpha+(-\rho+i\Omega)\alpha^*)+(-\rho+i\Omega)(\alpha-\alpha^*)\big]=\frac{2i\sqrt{\Omega}\alpha}{\sqrt{2\hbar\mu}}.\end{aligned}$$

Here, $\alpha_1=\operatorname{Re}\alpha$ and $\alpha_2=\operatorname{Im}\alpha$. Similarly, we have

$$-\frac{i\hbar}{2}\Big[\beta(0)\gamma(0)+\frac{-\rho+i\Omega}{\mu}\beta^2(0)\Big]=-\frac{i\hbar}{2}\frac{2i\sqrt{\Omega}\alpha}{\sqrt{2\hbar\mu}}\beta(0)=\frac{1}{2}\alpha\big[\alpha-\alpha^*\big]=\frac{1}{2}\big[\alpha^2-|\alpha|^2\big]=i\alpha\alpha_2. \tag{94}$$

Substituting (94) in (93), we obtain

$$\begin{aligned}\psi_a(x)&=\widehat{B}(0,\alpha,\mathbf{C}^1)\Phi_\nu(x,0,\mathbf{C}^1)=\\&=\exp\Big\{\frac{i\alpha\alpha_2}{2}\Big\}\exp\Big\{\frac{2i\sqrt{\Omega}}{\sqrt{2\hbar\mu}}\alpha x\Big\}\frac{i^\nu}{\sqrt{\nu!}}\Big(\frac{1}{\sqrt{2}}\Big)^\nu H_\nu\bigg(\sqrt{\frac{\Omega}{\hbar\mu}}\Big[x+\frac{\sqrt{2\hbar\mu}}{\Omega}\alpha_2\Big]\bigg)\times\\&\times\Big(\frac{1}{\pi\hbar}\Big)^{1/4}\Big(\frac{\Omega}{\mu}\Big)^{1/4}\exp\Big\{\frac{i}{2\hbar}\Big(\frac{-\rho+i\Omega}{\mu}\Big)x^2\Big\}.\end{aligned} \tag{95}$$

From (95), in particular, it follows that

$$|\psi_a(x)|^2 = \sqrt{\frac{\Omega}{\pi\hbar\mu}} \exp\Big\{ -\frac{\Omega}{\hbar\mu}\Big(x + \frac{\sqrt{2\hbar\mu}}{\sqrt{\Omega}}\alpha_2\Big)^2\Big\} \frac{1}{\nu!}\Big(\frac{1}{2}\Big)^\nu H_\nu^2\Big(\sqrt{\frac{\Omega}{\hbar\mu}}\Big[x + \frac{\sqrt{2\hbar\mu}}{\sqrt{\Omega}}\alpha_2\Big]\Big). \tag{96}$$

Similar to (85), we write Eqs. (25) determining the functionals $\mathbf{C}[\psi_a](t)$ for functions (95) as

$$\begin{aligned}
&X(0, C_1^1, C_2^1) = C_2^1 = x_\psi = -\frac{\sqrt{2\hbar\mu}}{\sqrt{\Omega}}\alpha_2, \quad P(0, C_1^1, C_2^1) = \frac{1}{\mu}(\bar{\Omega}C_1^1 - \rho C_2^1) = p_\psi = \frac{\sqrt{2\hbar\Omega}}{\sqrt{\mu}}\alpha_1,\\
&\Delta_{22}^{(2)}(0, \mathbf{C})) = (\Delta_{22}^{(2)})_\psi = \frac{\hbar}{2}(2\nu+1)\big|C(0)\big|^2 = \frac{\hbar\mu}{2\Omega}(2\nu+1),\\
&\Delta_{11}^{(2)}(0, \mathbf{C})) = (\Delta_{11}^{(2)})_\psi = \frac{\hbar}{2}\big|B(0)\big|^2(2\nu+1) = \frac{\hbar(\varrho^2+\Omega^2)}{2\Omega\mu}(2\nu+1),\\
&\Delta_{12}^{(2)}(0, \mathbf{C}) = (\Delta_{12}^{(2)})_\psi = \frac{\hbar}{4}\big[B(0)C^*(0) + B^*(0)C(0)\big](2\nu+1) = -\frac{\hbar\varrho}{2\Omega}(2\nu+1).
\end{aligned} \tag{97}$$

From equations (60) and (67), in view of (32), we obtain

$$\begin{gathered}
C_1^1 = C_1^1(\alpha) = \frac{\sqrt{2\hbar\mu\Omega}}{\bar{\Omega}}\Big(\alpha_1 - \frac{\rho}{\Omega}\alpha_2\Big), \quad C_2^1 = C_2^1(\alpha) = -\frac{\sqrt{2\hbar\mu}}{\sqrt{\Omega}}\alpha_2,\\
C_3^1 = C_4^1 = 0, \quad C_5^1 = \frac{\hbar\mu}{2\Omega}(2\nu+1).
\end{gathered} \tag{98}$$

Then the nonlinear symmetry operator for the nonlinear GPE (55), $\hat{A}(t, \alpha)$, determined by (72) and (91), transforms the solution $\Psi_\nu(x, t)$ (86) into a nonstationary solution $\widetilde{\Psi}_\nu(x, t)$:

$$\begin{aligned}
\widetilde{\Psi}_\nu(x, t, \alpha) &= (\hat{A}(t, \alpha)\Psi_\nu)(x, t) = \widehat{\widetilde{\mathcal{D}}}(t, \mathbf{C}^1, \mathbf{C}^0)\Big|_{C_5^0 = C_5^1 = (\hbar\mu/\Omega)(\nu+1/2),} \Psi_\nu(x, t) =\\
&= \frac{i^\nu}{\sqrt{\nu!}}\Big(\frac{1}{\sqrt{2}}\Big)^\nu\Big(\frac{1}{\pi\hbar}\Big)^{1/4}\Big(\frac{\Omega}{\mu}\Big)^{1/4} \exp\Big\{ -\frac{i}{2\hbar}\frac{\rho}{\mu}\Delta x^2 - \frac{1}{2\hbar}\frac{\Omega}{\mu}\Delta x^2\Big\} \times\\
&\times H_\nu\Big(\sqrt{\frac{\Omega}{\hbar\mu}}\Delta x\Big) \exp\Big\{ -i\Big(\nu + \frac{1}{2}\Big)\Big(\frac{\tilde{\varkappa}c\mu}{2\Omega} + \Omega\Big)t\Big\} \times\\
&\times \exp\Big\{\frac{i}{\hbar}[S(t, C_1^1, C_2^1) + P(t, C_1^1(\alpha), C_2^1(\alpha))\Delta x]\Big\},
\end{aligned} \tag{99}$$

which is localized around a phase orbit $\big(P\big(t, C_1^1(\alpha), C_2^1(\alpha)\big),\ X\big(t, C_1^1(\alpha), C_2^1(\alpha)\big)\big)$. Here $\Delta x = x - X(t, C_1^1(\alpha), C_2^1(\alpha))$ with the constants $C_1^1(\alpha)$, $C_2^1(\alpha)$ determined by equation (98), and a function $S(t, C_1^1(\alpha), C_2^1(\alpha))$ is determined by (70) where $C_1 = C_1^1(\alpha)$, $C_2 = C_2^1(\alpha)$, $C_3 = 0$, $C_4 = 0$, $C_5^0 = (\hbar\mu/\Omega)(\nu + 1/2)$.

In a linear case ($\varkappa = 0$), operators (99) with $\alpha \in \mathbb{C}$ constitute a representation of the Heisenberg–Weyl group [22,23]. The functions $\widetilde{\Psi}_\nu(x, t, \alpha)$ determined by (99) minimize the Schrödinger uncertainty relation for $\nu = 0$ [56], and, hence, they describe squeezed coherent states [57].

5 Discussion

Direct calculation of symmetry operators for nonlinear equations is, as a rule, a severe problem because of the nonlinearity and complexity of the determining equations [53].

However, for nearly linear equations [48], a wide class of symmetry operators can be constructed by solving linear determining equations for operators of this type much as symmetry operators are found for linear PDEs.

We have illustrated this situation by the example of the generalized multidimensional Gross–Pitaevskii equation (1). The formalism of semiclassical asymptotics leads to the semiclassically reduced GPE (6) (or (15)) which belongs to the class of nearly linear equations. Note that the solutions of the GPE are found in a special class of functions decreasing at infinity [47].

The reduced GPE is quadratic in the space coordinates and derivatives and contains a nonlocal term of special form.

In constructing the symmetry operators for the reduced Gross–Pitaevskii equation (15), we use the fact that this equation can be associated with the linear equation (20).

The symmetry operator $\hat{A}(t)$ (40) of the reduced GPE (15) has the structure of a linear pseudo-differential operator with coefficients $\mathbf{C}$ depending on the function Ψ on which the operator acts. The operator $\hat{A}$ is given in terms of the linear intertwining operator $\widehat{\mathcal{D}}$ and of the symmetry operators of the associated linear equation (20). The dependence of the coefficients $\mathbf{C}$ on Ψ arises from the algebraic condition (32), and therefore the operator $\hat{A}(t)$ is nonlinear. This is the key point of the approach.

The 1D examples considered show that for a special choice of the parameters $\mathbf{C}$, we can construct the symmetry operators and generate the families of solutions of the nonlinear equation (55) written in explicit form.

In conclusion, we note that the class of nearly linear equations can be considered as an approximation of highly nonlinear equations in the context of the semiclassical approximation formalism.

The further development of the study of symmetry operators is seen as a generalization of the approach for integro-differential GPE of more general form and for systems of equations of this type.

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